Journal of Pure and Applied Algebra 38 (1985) 313-322
North-Holland

THE LIE ALGEBRA STRUCTURE OF TANGENT COHOMOLOGY AND DEFORMATION THEORY

Michael SCHLESSINGER and James STASHEFF
Department of Mathematics, The University of North Carolina at Chapel Hill, Chapel Hill, NC 27514, USA

Communicated by C. Löfwall
Received 15 March 1985

Dedicated to Jan-Erik Roos on his 50-th birthday

Tangent cohomology of a commutative algebra is known to have the structure of a graded Lie algebra; we account for this by exhibiting a differential graded Lie algebra (in fact, two of them) equivalent as cochain complex to Harrison's yielding the tangent cohomology. This d.g. Lie algebra, called the tangent Lie algebra, also provides an interpretation of the cohomology in terms of perturbations of multiplicative resolutions and hence clarifies the relation to deformation theory. In particular, the higher order obstructions to deformations appear as Massey-Lie brackets. Moreover, we obtain homological constructions for the base and total spaces of a versal deformation.

Introduction

Harrison cohomology [5, 1962] for commutative algebras was defined as the analog of Hochschild cohomology for associative algebras. One of its most important applications is in the deformation theory of commutative algebras, where essential invariants belong to the Harrison groups $H^i(A,A)$ for $i=1,2,3$ [4, 1968]. These groups also appeared as the cohomology of the 'cotangent complex' of Lichtenbaum and Schlessinger [6, 1967] and, in characteristic zero, as the André-Quillen cohomology groups [2], [10]. We refer to this cohomology simply as *the tangent cohomology of the algebra.*

In Gerstenhaber's work and that of Nijenhuis [8], Harrison's cohomology (regraded) exhibits the structure of a graded Lie algebra but in rather ad hoc ways, not as the homology of a differential graded Lie algebra (d.g.l.). The same is true of André's construction [1] of the bracket, although André does use a method that is independent of the resolution in question of the algebra A.

Our main goal in this paper is to present Harrison (and André-Quillen) cohomology in characteristic zero as the homology of a natural d.g. Lie algebra associated to the augmented commutative algebra A, thus elucidating the d.g. Lie structure of the cohomology. This also substantiates the existence of Massey-Lie

0022-4049/85/$3.30

brackets in the cohomology, and we show the relevance of these cohomologies to the deformation theory of commutative algebras in a way that brings out the relevance of the Massey–Lie brackets.

Our basic point of view is that of multiplicative resolutions of the algebra A, i.e. a differential graded commutative algebra (c.d.g.a.) $\mathscr{A}$ together with a morphism of algebras $\mathscr{A} \to A$ inducing $H(\mathscr{A}) \approx A$. If $\mathscr{A}$ is free as a c.g.a., we call $\mathscr{A} \to A$ a *model.* For such resolutions, the deformations of A correspond to changes in the differential of $\mathscr{A}$; thus we were led to consider Der$\mathscr{A}$, the Lie algebra of graded derivations of $\mathscr{A}$. The original differential on $\mathscr{A}$ induces one on Der$\mathscr{A}$, making it a differential graded Lie algebra. The invariant analysis of deformation theory takes place conveniently in the tangent cohomology $T(A) = H$ (Der$\mathscr{A}$) although the homotopy type of Der $\mathscr{A}$ as d.g.l. is the true underlying invariant (cf. Palamodov [9] in the analytic case).

For comparison, Quillen considers only Der$(\mathscr{A}, A)$ the relative derivations of $\mathscr{A}$ into A regarding A as an $\mathscr{A}$-algebra via the original equivalence $\mathscr{A} \to A$. This fails to capture the Lie algebra structure, which is very important in the obstruction theory.

On the homology level, Der$\mathscr{A}$ and Der$(\mathscr{A}, A)$ agree and the latter homology was known to agree with the cohomology for commutative algebras constructed by Harrison [5]. We exhibit a particular model $\mathscr{A} \to A$ which makes the comparison with Harrison quite transparent, however at the expense of considering (differential graded) Lie *co*-algebras. This turns out to be the natural setting for Harrison's complex, which is precisely Hom$(\Gamma A, A)$ where ΓA is the free graded Lie coalgebra on the augmentation ideal of A considered to have degree 1. We then obtain a second computation for the tangent cohomology as the cohomology of the Lie algebra of 'outer derivations' of the construction ΓA; that is $H(s\Gamma A^* \# \mathrm{Der}\,\Gamma A) \approx T(A)$.

1. Algebras and coalgebras, commutative algebras and Lie coalgebras, resolutions and models

We work over a fixed ground field k of characteristic zero. We assume familiarity with associative (unitary) graded algebras (A, m), i.e. $A = \{A^n\}$ with m: $A^p \otimes A^q \to A^{p+q}$. We say A is commutative if $m(x,y) = (-1)^{pq} m(y,x)$ for $x \in A^p, y \in A^q$. A differential graded algebra (d.g.a.) (A, m, d) consists of a graded algebra (A, m) together with a graded differential d of degree 1 which is a derivation with respect to m, i.e. d: $A^p \to A^{p+1}$ with $d^2 = 0$ and $d(xy) = (dx)y + (-1)^{\deg x} x(dy)$, or $dm = m(d \otimes 1 + 1 \otimes d)$ where the signs are built into the definition of $f \otimes g$ for graded maps f and g, (i.e. $(f \otimes g)(x \otimes y) = (-1)^{(\deg g)(\deg x)} f(x) \otimes g(y)$).

Similarly we assume familiarity with the dual notion of associative (unitary) graded coalgebra (C, Δ), i.e. $C = \{C_n\}$ with $\Delta: C_n \to \sum_{p+q=n} C_p \otimes C_q$. We say C is commutative if $\Delta = T\Delta$, where $T(x \otimes y) = (-1)^{pq} y \otimes x$, where $x \in C_p$ and $y \in C_q$. A differential graded coalgebra (d.g.c.) (C, Δ, d) consists of a graded coalgebra (C, Δ) together with a graded differential d of degree -1 which is a coderivation with respect to Δ, i.e. $d: C_p \to C_{p-1}$ with $d^2 = 0$ and $\Delta d = (d \otimes 1 + 1 \otimes d)\Delta$.

Less familiar perhaps are the corresponding Lie notions. A *graded Lie algebra* $(L, [\cdot, \cdot])$ consists of a graded module $L = \{L^n\}$ together with a bilinear map $[\cdot, \cdot] : L^p \otimes L^q \to L^{p+q}$ such that for $x \in L^p$ and $y \in L^q$:

$$[x, y] = -(-1)^{pq}[y, x],$$

$$[x, [y, z]] = [[x, y], z] + (-1)^{pq}[y, [x, z]] \qquad \text{(the Jacobi identity).}$$

A *graded Lie coalgebra* (Γ, Δ) is a graded module $\Gamma = \{\Gamma_n\}$ with a diagonal $\Delta : \Gamma_n \to \sum_{p+q=n} \Gamma_p \otimes \Gamma_q$ such that

$$\Delta = -T\Delta,$$

$$(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta + (T \otimes 1)(1 \otimes \Delta)\Delta.$$

This can be summarized most succinctly and usefully by considering a suitable graded commutative algebra generated by Γ, as we shall soon see.

Free constructs. In each of the above categories, there exist free objects. Indeed any free graded k-module M gives rise to a corresponding free object satisfying the appropriate universal property in these categories.

The tensor algebra $T(M) = \{M^{\otimes n}\} = \{M \otimes \ldots \otimes M\}$ with

$$(a_1 \otimes \ldots \otimes a_p) \cdot (a_{p+1} \otimes \ldots \otimes a_n) = (a_1 \otimes \ldots \otimes a_n)$$

is the free (graded) associative algebra generated by M.

The tensor coalgebra $T^c(M) = \{M^{\otimes n}\}$ with

$$\Delta(a_1 \otimes \ldots \otimes a_n) = \sum_{p+q=n} (a_1 \otimes \ldots \otimes a_p) \otimes (a_{p+1} \otimes \ldots \otimes a_n)$$

is the free (graded) associative coalgebra cogenerated by $T^c(M) \to M$ [7], at least if M is of finite type.

The free Lie algebra $L(M) \subset T(M)$ can best be described by considering $T(M)$ as a Lie algebra with $[x, y] = x \circ y - (-1)^{(\deg x)(\deg y)} y \circ x$, where $\deg x = \sum \deg x_i$ for $x = x_1 \otimes \ldots \otimes x_n$, and similarly for y. Then $L(M)$ is the smallest sub Lie algebra containing M. Alternatively, consider $T(M)$ as a Hopf algebra primitively generated by M; i.e. $\Delta x = x \otimes 1 + 1 \otimes x$ for $x \in M$. This gives $T(M)$ the shuffle diagonal

$$\Delta(a_1 \otimes \ldots \otimes a_n)$$

$$= \sum_{\text{shuffles } \sigma} (-1)^{\sigma} (a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(p)}) \otimes (a_{\sigma(p+1)} \otimes \ldots \otimes a_{\sigma(n)})$$

where σ being a shuffle means $\sigma(1) < \sigma(2) < \ldots < \sigma(p)$ *and* $\sigma(p+1) < \ldots < \sigma(n)$ and $(-1)^{\sigma}$ means the sign of the shuffle of the graded elements. The free Lie algebra $L(M)$ is then the algebra of primitives $P(T(M))$, i.e. $x \in T(M)$ is primitive if and only if $\Delta x = x \otimes 1 + 1 \otimes x$.

The free Lie coalgebra can best be described as a quotient of $T^c(M)^+$, where +

denotes the part of strictly positive $\otimes$-degree. The tensor coalgebra $T^c(M)$ can be given a Hopf algebra structure by using the shuffle multiplication, i.e.

$$(a_1\otimes\ldots\otimes a_p)*(b_1\otimes\ldots\otimes b_q)=\sum(-1)^\sigma c_{\sigma(1)}\otimes\ldots\otimes c_{\sigma(p+q)}$$

where σ^{-1} is a shuffle permutation as above and $c_1\otimes\ldots\otimes c_{p+q}$ is just $a_1\otimes\ldots\otimes a_p\otimes b_1\otimes\ldots\otimes b_q$. The Lie coalgebra $L^c(M)$ consists of the indecomposables of this Hopf algebra, i.e. $T^c(M)^+/T^c(M)^+*T^c(M)^+$. Equivalently, $L^c(M)$ can be described as the quotient of $T^c(M)$ by the largest ideal in the kernel of $T^cM\to M$.

2. Harrison's cohomology as $H(s\Gamma A^*\#\,\mathrm{Der}\,\Gamma A)$

Let A be an augmented commutative algebra. Harrison [5] described his complex as follows: "Let E be an A-module. Let T be the tensor algebra (without the usual identity adjoined) of A. Then T with the shuffle product is a skew-commutative, graded algebra. $\mathrm{Hom}(T/T^2, E)$ will turn out to be a complex with the usual coboundary operator." The 'usual' coboundary operator is Hochschild's:

For f: $(T/T^2)_p\to E$, the coboundary df: $(T/T^2)_{p+1}\to E$ is given by

$$(df)(a_0,\ldots,a_p)==a_0f(a_1,\ldots,a_p)+\sum_{i=1}^{p-1}(-1)^if(a_0,\ldots,a_ia_{i+1},\ldots,a_p)$$

$$+(-1)^pa_pf(a_0,\ldots,a_{p-1}).$$

With hindsight, we can recognize T/T^2 as ΓA, the free Lie coalgebra on $s\bar{A}$, where $\bar{A}$ is the augmentation ideal of A and $(s\bar{A})^{p+1}=\bar{A}^p$. To analyze Harrison's construction further, consider that since ΓA is free as a Lie coalgebra over $s\bar{A}$, we can lift each homomorphism $\Gamma A\to\bar{A}$ to a coderivation α of ΓA; i.e., $\Delta\alpha=(\alpha\otimes 1+1\otimes\alpha)\Delta$, the signs being built into $1\otimes\alpha$ as usual. By abuse of notation, we refer to $\mathrm{Der}\,\Gamma A$, the graded *co*-derivations of ΓA. As graded k-modules, we then have

$$\mathrm{Hom}(\Gamma A,\bar{A})\approx\mathrm{Der}\,\Gamma A.$$

Since coderivations still compose via a bracket, $\mathrm{Der}\,\Gamma A$ is a d.g. Lie algebra and Gerstenhaber [4] has introduced a graded Lie bracket on $\mathrm{Hom}(\Gamma A,\bar{A})$. It is easy to check that the map to $\mathrm{Der}\,\Gamma A$ is a graded Lie map.

This analysis can be extended to all of $\mathrm{Hom}(\Gamma A, A)$ by adapting a construction due to Quillen [11] in a different context. As in [12], for any d.g. Lie coalgebra Γ, let $L=\mathrm{Hom}(\Gamma,k)$ as d.g. Lie algebra and let sL be the abelian Lie algebra with underlying vector space isomorphic to that of L with a shift in dimension. Define $sL\#\,\mathrm{Der}\,\Gamma$ to be the d.g. Lie semi-direct product of the abelian d.g. Lie algebra sL with $\mathrm{Der}\,\Gamma$ which acts on sL in the obvious way:

$$[sf,\theta]=s(f\circ\theta)$$

while

$$d(sf) = -sdf + \mathrm{ad}\, f$$

where ad f is the coderivation of Γ defined by the composite

$$\Gamma \xrightarrow{\Delta} \Gamma \otimes \Gamma \xrightarrow{f \otimes 1} k \otimes \Gamma \approx \Gamma.$$

If $\varepsilon : A \to k$ is an augmentation, i.e. $\varepsilon(1) = 1$, then ε induces a splitting of k-modules:

$$\mathrm{Hom}(\Gamma A, A) \approx \mathrm{Hom}(\Gamma A, k) \oplus \mathrm{Hom}(\Gamma A, \bar{A})$$

and thus we construct an isomorphism

$$\mathrm{Hom}(\Gamma A, A) \to sLA \# \mathrm{Der}\, \Gamma A, \quad \text{where } LA = \mathrm{Hom}(\Gamma A, k).$$

Gerstenhaber's specific formulas for $[\cdot, \cdot]$ on $\mathrm{Hom}(\Gamma A, A)$ correspond precisely to the Lie algebra structure on $sLA \# \mathrm{Der}\, \Gamma A$. Thus, as observed by Gerstenhaber, Harrison's cohomology $H(A, A) = H(sLA \# \mathrm{Der}\, \Gamma A)$ is a graded Lie algebra, except that Harrison's grading is shifted by one since he counts only the $\otimes$-grading, rather than the change in grading which is standard for Der. Finally we adopt the notation $T(A)$ for this cohomology with the Lie algebra grading: $T^i(A)$ for $i \geq 0$ is represented by derivations of degree $-i$, i.e. determined by $\theta : \bar{A}^{\otimes i+1} \to \bar{A}$. We refer to $T(A)$ as the tangent cohomology of A.

3. Multiplicative resolutions

A *multiplicative resolution* $\mathscr{A}$ of a commutative algebra A is a differential graded commutative algebra (c.d.g.a.) $\mathscr{A}$ together with a morphism of algebras $\mathscr{A} \to A$ inducing $H(\mathscr{A}) \approx A$. If $\mathscr{A}$ is free as a c.g.a., we call $\mathscr{A}$ a *model* for A.

The functor $\Gamma(\cdot)$ from commutative algebras to d.g. Lie coalgebras can be extended to the category of c.d.g.a.'s by incorporating d_A with the Hochschild coboundary $d_{\Gamma A}$: by extending d_A to ΓA as a coderivation with

$$d_A(sa) = -sd_A a$$

and adding it to $d_{\Gamma A}$.

An adjoint functor $\mathscr{A}$ from d.g. Lie coalgebras to c.d.g.a.'s can be constructed as an extension of Koszul's complex for defining Lie algebra cohomology:

Given a d.g. Lie coalgebra (Γ, Δ, d), define $\mathscr{A}(\Gamma)$ as the free commutative graded algebra $\mathrm{Sym}(s^{-1}\Gamma)$ on $s^{-1}\Gamma$ (where $s^{-1} : \Gamma^p \approx (s^{-1}\Gamma)^{p-1}$). That is,

$$\mathrm{Sym}(s^{-1}\Gamma) = T(s^{-1}\Gamma)/\Sigma = \{(s^{-1}\Gamma)^{\otimes n}/\Sigma_n\}$$

where the symmetric groups Σ_n act with the graded permutation signs. The diagonal $\Delta : \Gamma \to \Gamma \otimes \Gamma$ passes to $s^{-1}\Gamma \to \mathrm{Sym}^2(s^{-1}\Gamma)$ precisely because of the graded anticommutativity and the Jacobi identity is equivalent to

$$s^{-1}\Gamma \xrightarrow{\Delta} \mathrm{Sym}^2(s^{-1}\Gamma) \xrightarrow{\Delta} \mathrm{Sym}^3(s^{-1}\Gamma)$$

being zero, Δ being extended as a derivation to all of $\mathrm{Sym}(s^{-1}\Gamma)$. The total differential $D_{\mathscr{A}}$ is then $\Delta + d_\Gamma$ where d_Γ is the derivation determined by

$$d_\Gamma(s\Gamma) = -sd_\Gamma\Gamma.$$

The adjunction $\mathscr{A}\Gamma A \to A$ is a particularly useful model for A. Folllowing the convention in rational homotopy theory, we refer to ΓA as a *Lie model* for A and $\mathscr{A}\Gamma A \to A$ as a (c.d.g.a.) model for A.

For an augmented algebra A with trivial multiplication, i.e. $\mathfrak{m}^2 = 0$ where $\mathfrak{m} = \ker \varepsilon : A \to k$, the adjunction $\mathscr{A}\Gamma A \to A$ is the Tate resolution of k over A, [13].

4. Formal deformation theory

A formal one-parameter deformation of a commutative algebra A is a flat $k[[t]]$-algebra E such that $E/tE \approx A$, i.e. a flat extension

$$k[[t]] \to E \to A.$$

The deformation is trivial if, as algebras, $E \approx k[[t]] \otimes A$. Choosing a basis $\{b_i\}$ for A, the multiplication in E can be described in terms of structural power series $c_{ij}^k(t)$, i.e. $b_i b_j = \sum c_{ij}^k(t) b_k$. Assuming we are over a field (or A is free as k-module), then as $k[[t]]$-modules, we already have $E \approx k[[t]] \otimes A$; the twist is all in the multiplication. Thus to resolve E, we can use $k[[t]] \otimes \mathscr{A}$ where $\mathscr{A}$ is a multiplicative resolution of A, but with a differential $D = 1 \otimes d_{\mathscr{A}} + \sum t^i \otimes p_i$, where p_i is a derivation of $\mathscr{A}$ of degree 1. The condition $D^2 = 0$ implies a whole sequence of equations:

$$dp_1 + p_1 d = 0,$$

$$dp_2 + p_2 d + p_1 p_1 = 0,$$

etc.

Thus, in particular, D determines an 'infinitesimal tangent vector' $\theta = [p_1] \in H^1(\mathrm{Der}\,\mathscr{A}) = T^1(A)$.

Now we wish to start in $H^1(\mathrm{Der}\,\mathscr{A})$ and work back. Any class $\theta \in H^1(\mathrm{Der}\,\mathscr{A})$ determines a flat extension A_ε over the dual numbers $k[\varepsilon] = k[t]/(t^2)$. The obstruction to extending A to a flat extension over $k[t]/(t^3)$ is $[\theta, \theta] \in H^2(\mathrm{Der}\,\mathscr{A}) = T^2(A)$. That is, we need a representative p_1 of θ for which there is a p_2 with $dp_2 + p_2 d + p_1 p_1 = 0$. (Notice $p_1 p_1 = \frac{1}{2}[p_1, p_1]$.) More generally, the obstruction to extending a flat deformation over $k[t]/(t^n)$ to one over $k[t]/(t^{n+1})$ is a higher order Massey bracket $[\theta, \ldots, \theta]$ also lying in a quotient of $T^2(A)$. (Compare Nijenhuis [8] who has this except for the Massey bracket terminology.)

Massey brackets. The homology of any d.g. Lie algebra (L, d) has a system of higher order operations called Massey brackets, or rather two systems: general ones of n variables $[\theta_1, \ldots, \theta_n]$ and *restricted* ones of a single variable $[\theta, \ldots, \theta]$. We are interested only in the single variable system which is defined as follows:

For q odd and $\theta \in H^q(L)$, the *Massey triple bracket* $[\theta, \theta, \theta]$ is defined as a coset of $H^{3q-1}(L)$ if $[\theta, \theta] = 0$ as follows: Let $p \in L^q$ be a cycle in θ. Since $[\theta, \theta] = 0 \in H^{2q}(L)$, there is $b \in L^{2q-1}$ such that $db = [p, p]$. The Massey bracket is the set of homology classes of all the cycles of the form $[p, b]$ which is a coset mod $[\theta, H^{2q-1}(L)]$.

The generalization to higher order is formal but straight-forward. The *n-fold Massey bracket* $[\theta, \ldots, \theta]$ is

(1) defined if there exist $p_i \in L^{i(q-1)+1}$ such that p_1 represents θ and $\sum_{i+j=k} [p_i, p_j] = dp_k + p_k d$ for $2 \leq k < n$, and

(2) represented by $\sum_{i+j=n} [p_i, p_j]$ for all choices of p_i satisfying (1).

Of course our choice of notation has been rigged so that for $L = \operatorname{Der} \mathscr{A}$ and $q = 1$, $d + p_1 + \ldots + p_{n-1}$ determines a flat deformation of A over $k[t]/(t^n)$ and the n-fold bracket $[\theta, \ldots, \theta]$ contains zero if and only if A can be extended to a flat deformation over $k[t]/(t^{n+1})$. As usual with higher order operations, we are *not* attempting to extend a fixed deformation over $k[t]/(t^n)$ to one over $k[t]/(t^{n+1})$.

Remark. At the computational level, much of this can be carried out using ordinary projective (not multiplicative) resolutions, but the structures are somewhat obscured.

5. The tangent Lie algebra and deformation theory

Formal deformation theory can be described in terms of $\operatorname{Der} \mathscr{A}$ for any multiplicative resolution $\mathscr{A} \to A$, but it's often helpful to choose a special resolution, small for computations, large for theoretical comparisons. In particular, we can relate $\operatorname{Der} \mathscr{A}\Gamma A$ to the tangent Lie algebra: $sLA \# \operatorname{Der} \Gamma A$.

Theorem. *For any connected d.g. Lie coalgebra Γ, there is a map of d.g. Lie algebras*

$$\alpha : sL \# \operatorname{Der} \Gamma \to \operatorname{Der} \mathscr{A}\Gamma \quad \textit{where } L = \operatorname{Hom}(\Gamma, k)$$

which induces an isomorphism in homology (i.e. α is a weak homotopy equivalence of d.g. Lie algebras). Moreover $\alpha : sLA \# \operatorname{Der} \Gamma A \to \operatorname{Der} \mathscr{A}\Gamma A$ is a natural transformation of functors of c.d.g.a.'s.

Construction of α for $\Gamma = \Gamma A$. Since $\mathscr{A}\Gamma A$ is augmented (in fact, connected), we again have

$$\operatorname{Der} \mathscr{A}\Gamma A \# \operatorname{Hom}(s\Gamma A, \mathscr{A}\Gamma A) \approx \operatorname{Hom}(s\Gamma A, k) \oplus \operatorname{Hom}(S\Gamma A, \bar{\mathscr{A}}\Gamma A).$$

On the other hand, $\mathrm{Der}\,\Gamma A \approx \mathrm{Hom}(\Gamma A, \bar{A})$ maps naturally into $\mathrm{Hom}(s\Gamma A, \bar{\mathscr{A}}\Gamma A)$. The splitting used depends only on the augmentation of $\mathscr{A}$, so we obtain α, natural with respect to maps of augmented d.g. algebras. It is tedious but straightforward to check that α is a map of d.g. Lie algebras.

That α induces a homology isomorphism follows from the isomorphism of $sLA \# \mathrm{Der}\,\Gamma A$ with Harrison's complex $\mathrm{Hom}(\Gamma A, A)$ and the homology isomorphism with $\mathrm{Hom}(\Gamma A, \mathscr{A}\Gamma A) \approx \mathrm{Der}\,\mathscr{A}\Gamma A$ induced by the resolution $\mathscr{A}\Gamma A \to A$.

Corollary. *Let A be an augmented commutative algebra. Harrison and Quillen cohomologies agree for coefficients in any A-module M.*

With general coefficient A-module M, Harrison's cohomology is that of $\mathrm{Hom}(\Gamma A, M)$ and Quillen's is that of $\mathrm{Der}(\mathscr{A}\Gamma A, M)$ so the equivalence is straightforward. (An alternate approach to the comparison of Harrison cohomology with $\mathrm{Der}\,\mathscr{A}$ for a 'resolvent' $\mathscr{A}$ is given by Palamodov in the analytic setting over $\mathbb{C}$, based on ideas of Tjurina [9].)

In summary, the deformation theory of A can be described invariantly in terms of

$$T(A) \approx H(\mathrm{Der}\,\mathscr{A}\Gamma A) \approx H(sLA \# \mathrm{Der}\,\Gamma A);$$

the advantage of the last complex is its size compared to $\mathrm{Der}\,\mathscr{A}\Gamma A$ and its d.g. Lie structure compared to $\mathrm{Hom}(\Gamma A, A)$. This has proved computationally useful both in rational homotopy theory [12] and for special algebras, e.g. the 'thick point' $A = k[x_1, \dots, x_n]/(x_i x_j,\ 1 \le i \le j \le n)$, the trivial algebra whose maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$ has square zero. The Lie coalgebra ΓA is then free with $d = 0$. If $n > 1$, then $sLA \# \mathrm{Der}\,\Gamma A$ has the same homology as $\mathrm{Der}\,\Gamma A/\mathrm{ad}\Gamma A$ which has trivial differential, thus

$$T(A) \approx \mathrm{Der}\,L/\mathrm{ad}\,L$$

where L is the free Lie algebra on n variables (passing to the dual $L = \mathrm{Hom}(\Gamma A, k)$).

Now, every finite-dimensional k-algebra specializes to the trivial algebra A, by letting the multiplication degenerate. Thus the deformations of A consist of all commutative associative unitary multiplications on a vector space of dimension $n+1$, which is easily seen to be the set of degree 1 elements in $sL \# \mathrm{Der}\,L \simeq \mathrm{Der}\,L/\mathrm{ad}\,L$ which satisfy the equation $[\theta, \theta] = 0$. In other words, the base ring B of the so-called (mini) versal deformation of A is described symbolically as $\mathrm{Spec}\,B = \{\theta \in \mathrm{Der}^1 L/\mathrm{ad}\,L : [\theta, \theta] = 0\}$. (In other words, B is the quotient of the polynomial algebra on $\dim(\mathrm{Der}^1 L/\mathrm{ad}\,L)$ variables satisfying the indicated homogeneous quadratic equations.) In a similar way, augmented algebras of dimension $n+1$ correspond to $\theta \in \mathrm{Der}^1 L$ satisfying $[\theta, \theta] = 0$, and the mini-versal deformation ring E for augmented deformations of A is given by $\mathrm{Spec}\,E = \{\theta \in \mathrm{Der}^1 L : [\theta, \theta] = 0\}$.

The flat map $B \to E$ of rings, whose fibre $E/\mathfrak{m}_B E \approx A$, is the mini-versal deformation of A. Notice that Zariski tangent spaces $(\mathfrak{m}_B/\mathfrak{m}_B^2)^*$ and $(\mathfrak{m}_E/\mathfrak{m}_E^2)^*$ are respectively isomorphic to $T^1(A) = \mathrm{Der}\,L/\mathrm{ad}\,L$ and $\mathrm{Der}^1 L$. Note B (resp E) is not regular if $n>2$ (resp. 1): Their respective obstruction spaces $[M,M]$ ($M=\mathrm{Der}^1 L/\mathrm{ad}\, L$, resp. $\mathrm{Der}^1 L$) are not zero; however, in both cases cubic and higher order Massey-products vanish.

This example generalizes as follows. Given the augmented algebra A, we seek a flap map $B\to E$ of augmented algebras, with special fibre $E/\mathfrak{m}_B E=A$ (i.e. a 'deformation' of A which is 'versal' in the sense that it induces any other deformation $B'\to E'$ of A by a change of base 'classifying' morphism $B\to B'$. As it turns out, $E\to E\otimes_B E$ will then have a similar property for augmented deformations of A, i.e. pointed deformations of $\mathrm{Spec}\, A$.) Assuming, for convenience, that A is a finite-dimensional k-module, let L be $\mathrm{Hom}(\Gamma A, k)$, or any other Lie algebra model for A. We can then give a uniform description of the rings $R=A, B, E$ respectively as

$$\mathrm{Spec}\, R=\{\theta\in M^1 : d\theta+\tfrac{1}{2}[\theta,\theta]=0\}$$

where M is respectively L, $\mathrm{Der}\, L/\mathrm{ad}\, L\sim sL\#\mathrm{Der}\, L$ and $\mathrm{Der}\, L$. In particular $\mathrm{Der}\, L/\mathrm{ad}\, L$ and $\mathrm{Der}\, L$ are (non-free) Lie algebra models for B and E. Notice $B\to E$ is not mini versal (the dimension of M_B/M_B^2 is not the minimal possible one, namely $\dim T^1(A)$), unless it happens that the differential in the tangent algebra $\mathrm{Der}\, L/\mathrm{ad}\, L$ is zero, a condition called *formality* in the setting of rational homotopy theory. In this case, the cubic and higher order Massey product obstructions to the deformation of A vanish. In general this $B\to E$ may be thought of as a maximal versal deformation of A, in that the defining equations are (inhomogeneous) quadratic.

Thus, tangent cohomology $T(A)$ may be computed in either of two ways $T(A)=H(\mathrm{Der}\,\mathscr{A})=H(\mathrm{Der}\, L/\mathrm{ad}\, L)$ as the cohomology of a d.g. Lie algebra, where $\mathscr{A}$ (resp L) is an algebra (resp. Lie algebra) model for A. Whereas the cohomology $T^i(A)$ yields infinitesimal automorphisms ($i=0$), deformations ($i=1$), and deformation obstructions ($i=2$) of A, knowledge of the Lie algebra itself will yield a versal deformation of A.

We note that in case A is *graded,* the higher cohomology $T^i(A)$ may be interpreted topologically. If F is the formal topological space whose cohomology is A, then up to rational homotopy type, L is a Lie algebra model for F. It then follows [12] that the classifying space $B(\mathrm{Aut}\, F)$, the base space for the universal topological fibration with fibre F, has a Lie algebra model consisting of the non-positively graded part of either of the tangent Lie algebras $\mathrm{Der}\,\mathscr{A}\sim\mathrm{Der}\, L/\mathrm{ad}\, L$. It follows that the non-positively graded part of $T^i(A)$ may be identified with the rational homotopy $\pi_*(F)\otimes Q$.

References

[1] M. André, L'algèbre de Lie d'un anneau local, Symp. Math. IV (1970) 337–375.

[2] M. André, Méthode simpliciale en algèbre homologique et algèbre commutative, Lecture Notes in Math. 32 (Springer, Berlin, 1967).

[3] A. Douady, Obstruction primaire à la déformation, Seminaire Henri CARTAN (1960/61) exposé 4.

[4] M. Gerstenhaber, On the deformation of rings and algebras, III, Ann. Math. 88 (1968) 1–34.

[5] D.K. Harrison, Commutative algebra and cohomology, Trans. AMS 104 (1962) 191-204.
[6] S. Lichtenbaum and M. Schlessinger, The cotangent complex of a morphism, Trans. AMS 128 (1967) 41-70.
[7] W. Michaelis, Lie coalgebras, Adv. in Math. 38 (1980) 1-54.
[8] A. Nijenhuis, Graded Lie algebras and their applications (mimeographed), Math. Inst., Univ. van Amsterdam (1964), see also A. Nijenhuis and R.W. Richardson, Cohomology and deformation of algebraic structures, Bull. AMS 70 (1964) 406-411.
[9] V.P. Palamodov, Cohomology of analytic algebras, Trudy Moskow Math. Soc. 44 (1982) = Trans. Moscow Math. Soc. 44 (1983) 1-61.
[10] D. Quillen, On the (co-)homology of commutative rings, Proc. Symp. Pure Math. 17 (1970) 65-87.
[11] D. Quillen, Rational homotopy theory, Ann. Math. 90 (1969) 205-295.
[12] M. Schlessinger and J.D. Stasheff, Deformation theory and rational homotopy type, Publ. Math. IHES, to appear.
[13] J. Tate, Homology of Noetherian rings and local rings, Illinois J. Math. 1 (1957) 14-27.